

A note on the computation of Puiseux series solutions of the Riccati equation associated with a homogeneous linear ordinary differential equation

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Abstract

We present in this paper a detailed note on the computation of Puiseux series solutions of the Riccati equation associated with a homogeneous linear ordinary differential equation. This paper is a continuation of [1] which was on the complexity of solving arbitrary ordinary polynomial differential equations in terms of Puiseux series.

Introduction

Let $K = \mathbb{Q}(T_1, \dots, T_l)[\eta]$ be a finite extension of a finitely generated field over \mathbb{Q} . The variables T_1, \dots, T_l are algebraically independent over \mathbb{Q} and η is an algebraic element over the field $\mathbb{Q}(T_1, \dots, T_l)$ with the minimal polynomial $\phi \in \mathbb{Z}[T_1, \dots, T_l][Z]$. Let \overline{K} be an algebraic closure of K and consider the two fields:

$$L = \cup_{\nu \in \mathbb{N}^*} K((x^{\frac{1}{\nu}})), \quad \mathcal{L} = \cup_{\nu \in \mathbb{N}^*} \overline{K}((x^{\frac{1}{\nu}}))$$

which are the fields of fraction-power series of x over K (respectively \overline{K}), i.e., the fields of Puiseux series of x with coefficients in K (respectively \overline{K}). Each element $\psi \in L$ (respectively $\psi \in \mathcal{L}$) can be represented in the form $\psi = \sum_{i \in \mathbb{Q}} c_i x^i$, $c_i \in K$ (respectively $c_i \in \overline{K}$). The order of ψ is defined by $\text{ord}(\psi) := \min\{i \in \mathbb{Q}, c_i \neq 0\}$. The fields L and \mathcal{L} are differential fields with the differentiation

$$\frac{d}{dx}(\psi) = \sum_{i \in \mathbb{Q}} i c_i x^{i-1}.$$

Let $S(y) = 0$ be a homogeneous linear ordinary differential equation which is written in the form

$$S(y) = f_n y^{(n)} + \dots + f_1 y' + f_0 y$$

where $f_i \in K[x]$ for all $0 \leq i \leq n$ and $f_n \neq 0$ (we say that the order of $S(y) = 0$ is n). Let y_0, \dots, y_n be new variables algebraically independent over $K(x)$. We will associate to $S(y) = 0$ a non-linear differential polynomial $R \in K[x][y_0, \dots, y_n]$ such that y is a solution of

$S(y) = 0$ if and only if $\frac{y'}{y}$ is a solution of $R(y) = 0$ where the last equation is the ordinary differential equation $R(y, \frac{dy}{dx}, \dots, \frac{d^n y}{dx^n}) = 0$. We consider the change of variable $z = \frac{y'}{y}$, i.e., $y' = zy$, we compute the successive derivatives of y and we make them in the equation $S(y) = 0$ to get a non-linear differential equation $R(z) = 0$ which satisfies the above property. R is called the *Riccatti* differential polynomial associated with $S(y) = 0$. We will describe all the fundamental solutions (see e.g. [20, 13]) of the differential equation $R(y) = 0$ in \mathcal{L} by a differential version of the Newton polygon process. There is another way to formulate R : let $(r_i)_{i \geq 0}$ be the sequence of the following differential polynomials

$$r_0 = 1, \quad r_1 = y_0, \dots, r_{i+1} = y_0 r_i + D r_i, \quad \forall i \geq 1,$$

where $D y_i = y_{i+1}$ for any $0 \leq i \leq n-1$. We remark that for all $i \geq 1$, $r_i \in \mathbb{Z}[y_0, \dots, y_{i-1}]$ has total degree equal to i w.r.t. y_0, \dots, y_{i-1} and the only term of r_i of degree i is y_0^i .

Lemma 0.1 *The non-linear differential polynomial*

$$R = f_n r_n + \dots + f_1 r_1 + f_0 r_0 \in K[x][y_0, \dots, y_n]$$

is the Riccatti differential polynomial associated with $S(y) = 0$.

1 Newton polygon of R

Let R be the *Riccatti* differential polynomial associated with $S(y) = 0$ as in Lemma 0.1. We will describe the Newton polygon $\mathcal{N}(R)$ of R in the neighborhood of $x = +\infty$ which is defined explicitly in [1]. For every $0 \leq i \leq n$, mark the point $(\deg(f_i), i)$ in the plane \mathbb{R}^2 . Let \mathcal{N} be the convex hull of these points with the point $(-\infty, 0)$.

Lemma 1.1 *The Newton polygon of R in the neighborhood of $x = +\infty$ is \mathcal{N} , i.e., $\mathcal{N}(R) = \mathcal{N}$.*

Proof. For all $0 \leq i \leq n$, $\deg_{y_0, \dots, y_{i-1}}(r_i) = i$ and the only term of r_i of degree i is y_0^i , then $lc(f_i)x^{\deg(f_i)}y_0^i$ is a term of R and $\mathcal{N} \subset \mathcal{N}(R)$. For any other term of $f_i r_i$ in the form $b x^j y_0^{\alpha_0} \dots y_{i-1}^{\alpha_{i-1}}$, where $b \in K$, $j < \deg(f_i)$ and $\alpha_0 + \dots + \alpha_{i-1} < i$, the corresponding point $(j - \alpha_1 - \dots - (i-1)\alpha_{i-1}, \alpha_0 + \dots + \alpha_{i-1})$ is in the interior of \mathcal{N} . Thus $\mathcal{N}(R) \subset \mathcal{N}$. \square

Lemma 1.2 *For any edge e of $\mathcal{N}(R)$, the characteristic polynomial of R associated with e is a non-zero polynomial. For any vertex p of $\mathcal{N}(R)$, the indicial polynomial of R associated with p is a non-zero constant. Moreover, if the ordinate of p is i_0 , then $h_{(R,p)}(\mu) = lc(f_{i_0}) \neq 0$.*

Proof. By Lemma 1.1 each edge $e \in E(R)$ joints two vertices $(\deg(f_{i_1}), i_1)$ and $(\deg(f_{i_2}), i_2)$ of $\mathcal{N}(R)$. Moreover, the set $N(R, a(e), b(e))$ contains these two points. Then

$$0 \neq H_{(R,e)}(C) = lc(f_{i_1})C^{i_1} + lc(f_{i_2})C^{i_2} + t.$$

where t is a sum of terms of degree different from i_1 and i_2 . For any vertex $p \in V(R)$ of ordinate i_0 , $lc(f_{i_0})x^{\deg(f_{i_0})}y_0^{i_0}$ is the only term of R whose corresponding point p . Then

$$h_{(R,p)}(\mu) = lc(f_{i_0}) \neq 0. \square$$

Corollary 1.3 *For any edge $e \in E(R)$, the set $A_{(R,e)}$ is a finite set. For any vertex $p \in V(R)$, we have $A_{(R,p)} = \emptyset$.*

2 Derivatives of the Riccati equation

For each $i \geq 0$ and $k \geq 0$, the k -th derivative of r_i is the differential polynomial defined by

$$r_i^{(0)} := r_i, r_i^{(1)} := r'_i := \frac{\partial r_i}{\partial y_0} \text{ and } r_i^{(k+1)} := (r_i^{(k)})' = \frac{\partial^{k+1} r_i}{\partial y_0^{k+1}}.$$

Lemma 2.1 *For all $i \geq 1$, we have $r'_i = ir_{i-1}$. Thus for all $k \geq 0$, $r_i^{(k)} = (i)_k r_{i-k}$, where $(i)_0 := 1$ and $(i)_k := i(i-1) \cdots (i-k+1)$.*

Proof. We prove the first item by induction on i . For $i = 1$, we have $r'_1 = 1 = 1 \cdot r_0$. Suppose that this property holds for a certain i and prove it for $i + 1$. Namely,

$$\begin{aligned} r'_{i+1} &= (y_0 r_i + D r_i)' \\ &= y_0 r'_i + r_i + D r'_i \\ &= i y_0 r_{i-1} + r_i + D(i r_{i-1}) \\ &= i(y_0 r_{i-1} + D r_{i-1}) + r_i \\ &= i r_i + r_i \\ &= (i+1) r_i. \end{aligned}$$

The second item is just a result of the first one (by induction on k). \square

Definition 2.2 *Let R be the Riccati differential polynomial associated with $S(y) = 0$. For each $k \geq 0$, the k -th derivative of R is defined by*

$$R^{(k)} := \frac{\partial^k R}{\partial y_0^k} = \sum_{0 \leq i \leq n} f_i r_i^{(k)}.$$

Lemma 2.3 *For all $k \geq 0$, we have*

$$R^{(k)} = \sum_{0 \leq i \leq n-k} (i+k)_k f_{i+k} r_i.$$

Proof. For all $i < k$, we have $r_i^{(k)} = 0$ because $\deg_{y_0}(r_i) = i$. Then by Lemma 2.1, we get

$$\begin{aligned} R^{(k)} &= \sum_{k \leq i \leq n} f_i r_i^{(k)} \\ &= \sum_{k \leq i \leq n} f_i (i)_k r_{i-k} \\ &= \sum_{0 \leq j \leq n-k} (j+k)_k f_{j+k} r_j \end{aligned}$$

with the change $j = i - k$. \square

Corollary 2.4 *The k -th derivative of R is the Riccati differential polynomial of the following linear ordinary differential equation of order $n - k$*

$$S^{(k)}(y) := \sum_{0 \leq i \leq n-k} (i+k)_k f_{i+k} y^{(i)}.$$

Proof. By Lemmas 0.1 and 2.3. \square

3 Newton polygon of the derivatives of R

Let $0 \leq k \leq n$ and $R^{(k)}$ be the k -th derivative of R . In this subsection, we will describe the Newton polygon of $R^{(k)}$. Recall that $R^{(k)}$ is the k -th partial derivative of R w.r.t. y_0 , then by the section 2 of [1], the Newton polygon of $R^{(k)}$ is the translation of that of R defined by the point $(0, -k)$, i.e., $\mathcal{N}(R^{(k)}) = \mathcal{N}(R) + \{(0, -k)\}$. The vertices of $\mathcal{N}(R^{(k)})$ are among the points $(\deg(f_{i+k}), i)$ for $0 \leq i \leq n - k$ by Lemma 2.3. Then for each edge e_k of $\mathcal{N}(R^{(k)})$ there are two possibilities: the first one is that e_k is parallel to a certain edge e of $\mathcal{N}(R)$, i.e., e_k is the translation of e by the point $\{(0, -k)\}$. The second possibility is that the upper vertex of e_k is the translation of the upper vertex of a certain edge e of $\mathcal{N}(R)$ and the lower vertex of e_k is the translation of a certain point $(\deg(f_{i_0}), i_0)$ of $\mathcal{N}(R)$ which does not belong to e . In both possibilities, we say that the edge e is associated with the edge e_k .

Lemma 3.1 *Let $e_k \in E(R^{(k)})$ be parallel to an edge $e \in E(R)$. Then the characteristic polynomial of $R^{(k)}$ associated with e_k is the k -th derivative of that of R associated with e , i.e.,*

$$H_{(R^{(k)}, e_k)}(C) = H_{(R, e)}^{(k)}(C).$$

Proof. The edges e_k and e have the same inclination $\mu_e = \mu_{e_k}$ and $N(R^{(k)}, a(e_k), b(e_k)) = N(R, a(e), b(e)) + \{(0, -k)\}$. Then

$$\begin{aligned} H_{(R^{(k)}, e_k)}(C) &= \sum_{(\deg(f_{i+k}), i) \in N(R^{(k)}, a(e_k), b(e_k))} (i+k)_k l c(f_{i+k}) C^i \\ &= \sum_{(\deg(f_j), j) \in N(R, a(e), b(e))} (j)_k l c(f_j) C^{j-k} \\ &= H_{(R, e)}^{(k)}(C). \square \end{aligned}$$

Corollary 3.2 *For any edge $e_k \in E(R^{(k)})$, the set $A_{(R^{(k)}, e_k)}$ is a finite set, i.e., $H_{(R^{(k)}, e_k)}(C)$ is a non-zero polynomial. For any vertex $p_k \in V(R^{(k)})$, we have $A_{(R^{(k)}, p_k)} = \emptyset$.*

Proof. By Corollaries 2.4 and 1.3. \square

4 Newton polygon of evaluations of R

Let R be the *Riccati* differential polynomial associated with $S(y) = 0$. Let $0 \leq c \in \overline{K}$, $\mu \in \mathbb{Q}$ and $R_1(y) = R(y + cx^\mu)$. We will describe the Newton polygon of R_1 for different values of c and μ .

Lemma 4.1 *R_1 is the Riccati differential polynomial of the following linear ordinary differential equation of order less or equal than n*

$$S_1(y) := \sum_{0 \leq i \leq n} \frac{1}{i!} R^{(i)}(cx^\mu) y^{(i)}.$$

Proof. It is equivalent to prove the following analogy of Taylor formula:

$$R_1 = \sum_{0 \leq i \leq n} \frac{1}{i!} R^{(i)}(cx^\mu) r_i$$

which is proved in Lemma 2.1 of [13]. \square

Then the vertices of $\mathcal{N}(R_1)$ are among the points $(\deg(R^{(i)}(cx^\mu), i)$ for $0 \leq i \leq n$. Thus the Newton polygon of R_1 is given by (Lemma 2.2 of [13]):

Lemma 4.2 *If μ is the inclination of an edge e of $\mathcal{N}(R)$, then the edges of $\mathcal{N}(R_1)$ situated above e are the same as in $\mathcal{N}(R)$. Moreover, if c is a root of $H_{(R,e)}$ of multiplicity $m > 1$ then $\mathcal{N}(R_1)$ contains an edge e_1 parallel to e originating from the same upper vertex as e where the ordinate of the lower vertex of e_1 equals to m . If $m = \deg H_{(R,e)}$, then $\mathcal{N}(R_1)$ contains an edge with inclination less than μ originating from the same upper vertex as e .*

Remark 4.3 *If we evaluate R on cx^μ we get*

$$R(cx^\mu) = \sum_{0 \leq i \leq n} f_i \times (c^i x^{i\mu} + t),$$

where t is a sum of terms of degree strictly less than $i\mu$. Then

$$lc(R(cx^\mu)) = \sum_{i \in B} lc(f_i) c^i = \sum_{(\deg(f_i), i) \in e} lc(f_i) c^i = H_{(R,e)}(c),$$

where

$$\begin{aligned} B &:= \{0 \leq i \leq n; \deg(f_i) + i\mu = \max_{0 \leq j \leq n} (\deg(f_j) + j\mu; f_j \neq 0)\} \\ &= \{0 \leq i \leq n; (\deg(f_i), i) \in e \text{ and } f_i \neq 0\}. \end{aligned}$$

Lemma 4.4 *Let μ be the inclination of an edge e of $\mathcal{N}(R)$ and c be a root of $H_{(R,e)}$ of multiplicity $m > 1$. Then*

$$H_{(R_1, e_1)}(C) = H_{(R,e)}(C + c)$$

where e_1 is the edge of $\mathcal{N}(R_1)$ given by Lemma 4.2. In addition, if e' is an edge of $\mathcal{N}(R_1)$ situated above e (which is also an edge of $\mathcal{N}(R)$ by Lemma 4.2) then $H_{(R_1, e)}(C) = H_{(R,e)}(C)$.

Proof. We have

$$\begin{aligned} H_{(R,e)}(C + c) &= \sum_{m \leq k \leq n} \frac{1}{k!} H_{(R,e)}^{(k)}(c) C^k \\ &= \sum_{m \leq k \leq n} \frac{1}{k!} H_{(R^{(k)}, e)}(c) C^k \\ &= \sum_{m \leq k \leq n} \frac{1}{k!} lc(R^{(k)}(cx^\mu)) C^k \\ &= H_{(R_1, e_1)}(C) \end{aligned}$$

where the first equality is just the Taylor formula taking into account that c is a root of $H_{(R,e)}$ of multiplicity $m > 1$. The second equality holds by Lemma 3.1. The third one by Remark 4.3. The fourth one by Lemma 4.1 and by the definition of the *characteristic* polynomial. \square

5 Application of Newton-Puiseux algorithm to R

We apply the Newton-Puiseux algorithm described in [1] to the *Riccatti* differential polynomial R associated with the linear ordinary differential equation $S(y) = 0$. This algorithm constructs a tree $\mathcal{T} = \mathcal{T}(R)$ with a root τ_0 . For each node τ of \mathcal{T} , it computes a finite field $K_\tau = K[\theta_\tau]$, elements $c_\tau \in K_\tau$, $\mu_\tau \in \mathbb{Q} \cup \{-\infty, +\infty\}$ and a differential polynomial R_τ as above. Let \mathcal{U} be the set of all the vertices τ of \mathcal{T} such that either $\deg(\tau) = +\infty$ and for the ancestor τ_1 of τ it holds $\deg(\tau_1) < +\infty$ or $\deg(\tau) < +\infty$ and τ is a leaf of \mathcal{T} . There is a bijective correspondance between \mathcal{U} and the set of the solutions of $R(y) = 0$ in the differential field \mathcal{L} . The following lemma is a differential version of Lemma 2.1 of [4] which separates any two different solutions in \mathcal{L} of the *Riccatti* equation $R(y) = 0$.

Lemma 5.1 *Let $\psi_1, \psi_2 \in \mathcal{L}$ be two different solutions of the differential Riccatti equation $R(y) = 0$. Then there exist an integer $\gamma = \gamma_{12}$, $1 \leq \gamma < n$, elements $\xi_1, \xi_2 \in \overline{K}$, $\xi_1 \neq \xi_2$ and a number $\mu_{12} \in \mathbb{Q}$ such that*

$$\text{ord}(R^{(\gamma)}(\psi_i) - \xi_i x^{\mu_{12}}) < \mu_{12}, \text{ for } i = 1, 2.$$

Proof. By the above bijection, there are two elements u_1 and u_2 of \mathcal{U} which correspond respectively to ψ_1 and ψ_2 . Let $i_0 = \max\{i \geq 0; \tau_i(u_1) = \tau_i(u_2)\}$. Denote by $\tau := \tau_{i_0}(u_1) = \tau_{i_0}(u_2)$ and $\tau_1 := \tau_{i_0+1}(u_1)$, $\tau_2 := \tau_{i_0+1}(u_2)$. We have $\tau_1 \neq \tau_2$ and $\epsilon := \max(\mu_{\tau_1}, \mu_{\tau_2})$ is the inclination of a certain edge e of $\mathcal{N}(R_\tau)$. There are three possibilities for ϵ :

- If $\mu_{\tau_2} < \mu_{\tau_1}$ then $\epsilon = \mu_{\tau_1} = \mu_e$. We have c_{τ_1} is a root of $H_{(R_\tau, e)}$ of multiplicity $m_1 \geq 1$ and $R_{\tau_1} = R_\tau(y + c_{\tau_1} x^{\mu_{\tau_1}})$. Then by Lemma 4.2 there is an edge e_1 of $\mathcal{N}(R_{\tau_1})$ parallel to e (so its inclination is $\epsilon = \mu_{\tau_1}$) originating from the same upper vertex as e where the ordinate of the lower vertex of e_1 equals to m_1 . In addition, e is also an edge of $\mathcal{N}(R_{\tau_2})$ and by Lemma 4.4, we have $H_{(R_{\tau_2}, e)}(C) = H_{(R_\tau, e)}(C)$ and

$$H_{(R_{\tau_1}, e_1)}(C) = H_{(R_\tau, e)}(C + c_{\tau_1}). \quad (1)$$

- If $\mu_{\tau_1} < \mu_{\tau_2}$ then $\epsilon = \mu_{\tau_2}$. Then by Lemma 4.2 there is an edge e_2 of $\mathcal{N}(R_{\tau_2})$ parallel to e originating from the same upper vertex as e . By Lemma 4.4, we have $H_{(R_{\tau_1}, e)}(C) = H_{(R_\tau, e)}(C)$ and

$$H_{(R_{\tau_2}, e_2)}(C) = H_{(R_\tau, e)}(C + c_{\tau_2}). \quad (2)$$

- If $\mu_{\tau_1} = \mu_{\tau_2} = \epsilon$ then c_{τ_1} and c_{τ_2} are two distinct roots of the same polynomial $H_{(R_\tau, e)}(C)$. Then equalities of type (1) and (2) hold.

Set $\gamma = \deg_C(H_{(R_\tau, e)}) - 1 \leq \deg_{y_0, \dots, y_n}(R) - 1 \leq n - 1 < n$ and $\gamma \geq 1$ because that the polynomial $H_{(R_\tau, e)}(C)$ has at least two distinct roots c_{τ_1} and c_{τ_2} . Moreover, we have $\text{ord}(\psi_i - y_{\tau_i}) < \epsilon$ for $i = 1, 2$. Let $\bar{\xi}_1 \in K_{\tau_1}$ and $\bar{\xi}_2 \in K_{\tau_2}$ be the coefficients of C^γ in the expansion of $H_{(R_{\tau_1}, e_1)}(C)$ and $H_{(R_{\tau_2}, e_2)}(C)$ respectively. There is a point (μ_{12}, γ) on the edge e which corresponds to the term of $H_{(R_\tau, e)}(C)$ of degree γ . We know by Lemma 4.1 that

$$R(y + \psi_i) = \sum_{0 \leq j \leq n} \frac{1}{j!} R^{(j)}(\psi_i) r_j \text{ for } i = 1, 2.$$

Then $\text{ord}(R^{(\gamma)}(\psi_i) - \gamma! \bar{\xi}_i x^{\mu_{12}}) < \mu_{12}$ for $i = 1, 2$. This proves the lemma by taking $\xi_i = \gamma! \bar{\xi}_i$ for $i = 1, 2$. \square

Let $\{\Psi_1, \dots, \Psi_n\}$ be a fundamental system of solutions of the linear differential equation $S(y) = 0$ (see e.g. [20, 10, 13]) and ψ_1, \dots, ψ_n be their logarithmic derivatives respectively, i.e., $\psi_1 = \Psi_1'/\Psi_1, \dots, \psi_n = \Psi_n'/\Psi_n$. Then $R(\psi_i) = 0$ for all $1 \leq i \leq n$.

Definition 5.2 Let ψ be an element of the field \mathcal{L} . We denote by $\text{span}_r(\psi)$ the r -differential span of ψ , i.e., $\text{span}_r(\psi)$ is the \mathbb{Z} -module generated by $r_1(\psi), r_2(\psi), \dots$.

Lemma 5.3 Let $\psi \in \mathcal{L}$ be a solution of a Riccati equation $R_2(y) = 0$ where $R_2 \in \mathbb{Z}[y_0, \dots, y_n]$ of degree n . Then $\text{span}_r(\psi)$ is the \mathbb{Z} -module generated by $r_1(\psi), \dots, r_{n-1}(\psi)$.

Proof. Write R_2 in the form $R_2 = r_n + \alpha_{n-1}r_{n-1} + \dots + \alpha_1r_1 + \alpha_0$ where $\alpha_i \in \mathbb{Z}$ for all $0 \leq i < n$. Then

$$\begin{aligned} r_{n+1}(\psi) &= \psi r_n(\psi) + Dr_n(\psi) \\ &= \sum_{0 \leq i < n} \alpha_i (\psi r_i(\psi) + Dr_i(\psi)) \\ &= \sum_{0 \leq i < n} \alpha_i r_{i+1}(\psi) \\ &= \sum_{0 \leq i < n} \beta_i r_i(\psi) \end{aligned}$$

for suitable $\beta_i \in \mathbb{Z}$ using the fact that $r_n(\psi) = \sum_{0 \leq i < n} \alpha_i r_i(\psi)$. \square

Consider a \mathbb{Z} -module $M := \text{span}_r(\psi_1, \dots, \psi_n)$, i.e., M is the \mathbb{Z} -module generated by $r_1(\psi_i), r_2(\psi_i), \dots$ for all $1 \leq i \leq n$. We define now what we call a r -cyclic vector for M (this definition is similar to that of the cyclic vectors in [16, 9, 19]).

Definition 5.4 An element $m \in M$ is called a r -cyclic vector for M if $M = \text{span}_r(m)$.

The following theorem is called a r -cyclic vector theorem. It is similar to the cyclic vector theorem of [15, 16, 9, 19].

Theorem 5.5 Let M be the \mathbb{Z} -module defined as above. There is a r -cyclic vector m for M .

Corollary 5.6 Let $m \in M$ be a r -cyclic vector for M . Then for any $1 \leq i \leq n$, there exists a Riccati differential polynomial $R_i \in \mathbb{Z}[y_0, \dots, y_n]$ such that $\psi_i = R_i(m)$.

Lemma 5.7 For each element $m \in M$, one can compute a Riccati differential polynomial $R_m \in K[x][y_0, \dots, y_n]$ such that $R_m(m) = 0$. In addition, there is a positive integer s such that the order of $R_m(y) = 0$ and the degree of R_m w.r.t. y_0, \dots, y_n are $\leq n^s$.

Proof. Each element $m \in M$ has the form $m = \alpha_1\psi_1 + \dots + \alpha_n\psi_n$ where $\alpha_1, \dots, \alpha_n \in \mathbb{Q}$. Then

$$m = \frac{(\Psi_1^{\alpha_1} \dots \Psi_n^{\alpha_n})'}{\Psi_1^{\alpha_1} \dots \Psi_n^{\alpha_n}}$$

is the logarithmic derivative of $\Psi_1^{\alpha_1} \dots \Psi_n^{\alpha_n}$. Or Lemma 3.8 (a) of [17] (see also [18, 19]) proves that one can construct a linear differential equation $S_m(y) = 0$, denoted by

$S^{\otimes \alpha_1 + \dots + \alpha_n}(y) = 0$ of order $\leq n^{\alpha_1 + \dots + \alpha_n}$ such that $\Psi_1^{\alpha_1} \dots \Psi_n^{\alpha_n}$ is one of its solutions. The equation $S^{\otimes \alpha_1 + \dots + \alpha_n}(y) = 0$ is called the $(\alpha_1 + \dots + \alpha_n)$ -th symmetric power of the linear differential equation $S(y) = 0$. In order to compute the equation $S_m(y) = 0$ associated with the linear combination $m = \alpha_1 \psi_1 + \dots + \alpha_n \psi_n \in M$, we take the change of variable $z = y^{\alpha_1 + \dots + \alpha_n}$ where y is a solution of $S(y) = 0$ and we compute the successive derivatives of z until we get a linear dependent family over K . The relation between these successive derivatives gives us the linear differential equation $S_m(z) = 0$. Let R_m be the *Riccatti* differential polynomial associated with $S_m(y) = 0$, then m is a solution of the equation $R_m(y) = 0$. \square

Remark 5.8 For any $1 \leq i \leq n$, we can take $R_{\psi_i} = R$ where R_{ψ_i} is defined in Lemma 5.7.

References

- [1] A. Ayad, *On the complexity of solving ordinary differential equations in terms of Puiseux series*, Preprint IRMAR (Institut de Recherche Mathématique de Rennes), May 2007. See <http://arxiv.org/abs/0705.2127>
- [2] J. Cano, *The Newton Polygon Method for Differential Equations*, Computer Algebra and Geometric Algebra with Applications, 2005, p. 18-30.
- [3] J. Cano, *On the series defined by differential equations, with an extension of the Puiseux Polygon construction to these equations*, International Mathematical Journal of Analysis and its Applications, 13, 1993, p. 103-119.
- [4] A. Chistov, *Polynomial Complexity of the Newton-Puiseux Algorithm*, Mathematical Foundations of Computer Science 1986, p. 247 - 255.
- [5] A. Chistov, D. Grigoriev, *Polynomial-time factoring of the multivariable polynomials over a global field*, Preprint LOMI E-5-82, Leningrad, 1982.
- [6] A.L. Chistov, D. Grigoriev, *Subexponential-time solving systems of algebraic equations*, I and II, LOMI Preprint, Leningrad, 1983, E-9-83, E-10-83.
- [7] A.L. Chistov, *Algorithm of polynomial complexity for factoring polynomials and finding the components of varieties in subexponential time*, J. Sov. Math., 34(1986), No. 4 p. 1838-1882.
- [8] A.L. Chistov, *Polynomial complexity algorithms for computational problems in the theory of algebraic curves*, Journal of Mathematical Sciences, 59 (3), 1992, p. 855-867.
- [9] R. C. Churchill, J. J. Kovacic, *Cyclic Vectors*, Article submitted to Journal of Symbolic Computation.
- [10] J. Della Dora, G. Di Crescenzo, E. Tournier, *An Algorithm to Obtain Formal Solutions of a Linear Homogeneous Differential Equation at an Irregular Singular Point*, EUROCAM 1982, p. 273-280.
- [11] J. Della Dora, F. Richard-Jung, *About the Newton algorithm for non-linear ordinary differential equations*, Proceedings of the 1997 international symposium on Symbolic and algebraic computation, United States, p. 298 - 304.

- [12] D. Grigoriev, *Factorization of polynomials over a finite field and the solution of systems of algebraic equations*, J. Sov. Math., 34(1986), No.4 p. 1762-1803.
- [13] D. Grigoriev, *Complexity of factoring and GCD calculating of ordinary linear differential operators*, J. Symp. Comput., 1990, vol.10, N 1, p. 7-37.
- [14] D. Grigoriev, M. F. Singer, *Solving ordinary differential equations in terms of series with real exponents*, Trans. AMS, 1991, vol. 327, N 1, p. 329-351.
- [15] N. Katz *A simple algorithm for cyclic vectors*, Amer. J. Math. 109 (1987), p. 65-70.
- [16] J. J. Kovacic, *Cyclic vectors and Picard-Vessiot extensions*, Technical report, Prolifics, Inc., 1996.
- [17] M. F. Singer, *Liouvillian Solutions of n -th Order Homogeneous Linear Differential Equations*, Am. J. Math., 103(4), 1981, p. 661-682.
- [18] M. F. Singer, F. Ulmer, *Galois Groups of Second and Third Order Linear Differential Equations*, Journal of Symbolic Computation, 16, July 1993, p. 9 - 36.
- [19] M. van der Put, M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, Volume 328, Springer, 2003.
- [20] W. Wasow, *Asymptotic Expansions for Ordinary Differential Equations*, New York, Kreiger Publ. Co. 1976.